HORTON LAW IN SELF-SIMILAR TREES

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Horton laws, which are akin to power-law distribution of the element sizes in a branching system, epitomize the scale invariance of natural dendritic structures. It is very intuitive that the existence of Horton laws should be related to the self-similar organization of branching, defined in suitable terms. Such relation, however, has escaped a rigorous explanation, remaining for long time a part of science literature folklore. This paper shows that a weak (mean) invariance under the operation of tree pruning is sufficient for the Horton law of branch numbers to hold in the strongest sense, hence explaining and unifying many earlier empirical observations and partial results in this direction. In particular, the existence of the Horton laws in Tokunaga self-similar trees is a special case of our general result.

Technically, self-similarity of random trees is related to the operation of pruning. Pruning $\mathcal{R}$ cuts the leaves and their parental edges and removes the resulting chains of degree-two nodes from a finite tree. A Horton-Strahler order of a vertex v and its parental edge is defined as the minimal number of prunings necessary to eliminate the subtree rooted at v. A branch is a group of neighboring vertices and edges of the same order. The Horton numbers $\mathcal{N}_k[K]$ and $\mathcal{N}_{ij}[K]$ are defined as the expected number of branches of order k, and the expected number of order-i branches that merged order-j branches, $j > i$, respectively, in a finite tree of order K. The Tokunaga coefficients are defined as $T_{ij}[K] = \mathcal{N}_{ij}[K]/\mathcal{N}_j[K]$. The pruning decreases the orders of tree vertices by unity. A rooted full binary tree is said to be mean-self-similar if its Tokunaga coefficients are invariant with respect to pruning: $T_k := T_{i,i+k}[K]$. We show that for self-similar trees, the condition $\limsup_{k\to\infty} T_k^{1/k} < \infty$ is necessary and sufficient for the existence of the strong Horton law: $\mathcal{N}_k[K]/\mathcal{N}_1[K] \to R^{1-k}$, as $K \to \infty$ for some $R > 0$ and every $k \geq 1$.